

HAMILTONIAN L-STABILITY OF LAGRANGIAN TRANSLATING SOLITONS

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ABSTRACT. In this paper, we compute the first and second variation formulas for the F-functional of translating solitons and study the Hamiltonian L-stability of Lagrangian translating solitons. We prove that any Lagrangian translating soliton is Hamiltonian L-stable.

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1. INTRODUCTION

An n -dimensional submanifold Σ^n of \mathbf{R}^{n+p} is called a self-shrinker if it is the time $t = -1$ slice of a self-shrinking mean curvature flow that disappears at $(0, 0)$, i.e. of a mean curvature flow satisfying $\Sigma_t = \sqrt{-t}\Sigma_{-1}$. We can also consider a self-shrinker as a submanifold that satisfies

$$H = -\frac{1}{2}x^\perp.$$

An n -dimensional submanifold Σ^n of \mathbf{R}^{n+p} is called a translating soliton if there is a constant vector T so that $\Sigma_t = \Sigma + tT$ is a solution to the mean curvature flow. We can also consider a translating soliton as a submanifold that satisfies

$$H = T^\perp.$$

According to the blow up rate of the second fundamental form, Huisken [6] classified the singularities of mean curvature flows into two types: Type I and Type II. Any Type I singularity of the mean curvature flow must be a self-shrinker ([6]). Type II singularity is one class of eternal solutions, which is defined for $-\infty < t < \infty$. One of the most important example of Type II singularity is the translating soliton ([5, 7]).

In this paper, we mainly study the stability (in some sense) of translating solitons. It was motivated by the work of Colding-Minicozzi [4], where they introduced the concept of F-stability of a self-shrinker in the hypersurface case. The definitions of many concepts in their paper can be naturally generalized to the higher codimension case (cf. [1, 2, 8]).

Given $x_0 \in \mathbf{R}^{n+p}$ and $t_0 > 0$, F_{x_0, t_0} is defined by

$$F_{x_0, t_0}(\Sigma) = (4\pi t_0)^{-\frac{n}{2}} \int_{\Sigma} e^{-\frac{|x-x_0|^2}{4t_0}} d\mu.$$

In [4], Colding-Minicozzi proved that self-shrinkers are the critical points for the $F_{0,1}$ functional by computing the first variation formula of $F_{0,1}$. They also computed the second

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variation formula, and defined F-stability of a self-shrinker by modding out translations. They showed that the round sphere and hyperplanes are the only F-stable self-shrinkers in \mathbf{R}^{n+1} .

In 2002, Andrews-Li-Wei [1], Arezzo-Sun [2] and Lee-Lue [8] independently generalized some of Colding-Minicozzi's work [4] from the hypersurface case to the higher codimensional case. They computed the first and second variation formulas for the F-functional, and studied F-stability of self-shrinkers in higher codimension.

Recently, motivated by an observation by Oh[10], Li-Zhang[9] and the author [12] studied Lagrangian F-stability and Hamiltonian F-stability of Lagrangian self-shrinkers, and proved characterization theorems for Hamiltonian F-stability of Lagrangian self-shrinkers, which characterize the Hamiltonian F-stability by the eigenvalues and eigenspaces of the drifted Laplacian.

With the above known results for self-shrinkers, it is natural to think that translating solitons might also have some similar properties. In fact, translating solitons are also critical points for an F -functional, which was studied by some people (See [3], [11] and [13] for example). The F -functional is defined by

$$F(\Sigma) = \int_{\Sigma} e^{\langle T, x \rangle} d\mu.$$

It is not as good as in the self-shrinker case because here $F(\Sigma)$ is usually infinity if Σ is a translating soliton, since any translating soliton is noncompact and $e^{\langle T, x \rangle} \rightarrow \infty$ very quickly as $x \rightarrow \infty$. This makes it hard to get many corresponding results as in the self-shrinker case.

However, if we require variation vector fields to have compact support, we can still compute the first and second variation formulas and consider stability of translating solitons. In [11], Shahriyari defined L-stability of translating surfaces in \mathbf{R}^3 , and proved that any translating graph in \mathbf{R}^3 is L-stable.

Especially, if a translating soliton is also a Lagrangian submanifold of the Euclidean space, we call it a Lagrangian translating soliton. We consider Hamiltonian L-stability (see section 2 for the definition) of Lagrangian translating solitons. Our main theorem is

Theorem 1.1. *Any Lagrangian translating soliton is Hamiltonian L-stable.*

Since we have this theorem, an interesting question is that whether this theorem has some application that could help us study the Lagrangian mean curvature flow. Besides, we are also interested in the L-stability of translating solitons in the hypersurface case.

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2. VARIATION FORMULAS AND HAMILTONIAN L-STABILITY

2.1. First and second variation formulas. Recall that the F -functional is defined by

$$F(\Sigma) = \int_{\Sigma} e^{\langle T, x \rangle} d\mu.$$

The first variation formula of F is

Lemma 2.1. *Let $\Sigma_s \subset \mathbf{R}^{n+p}$ be a compactly supported variation of Σ with normal variation vector field V , then*

$$(2.1) \quad \frac{\partial}{\partial s}(F(\Sigma_s)) = \int_{\Sigma} \langle T^{\perp} - H, V \rangle e^{\langle T, x \rangle} d\mu_{\Sigma}.$$

Proof. From the first variation formula (for area), we know that

$$(d\mu)' = -\langle H, V \rangle d\mu.$$

It follows that

$$\frac{\partial}{\partial s}(F(\Sigma_s)) = \int_{\Sigma} e^{\langle T, x \rangle} \langle T, V \rangle d\mu - \int_{\Sigma} e^{\langle T, x \rangle} \langle H, V \rangle d\mu = \int_{\Sigma} \langle T^{\perp} - H, V \rangle e^{\langle T, x \rangle} d\mu.$$

This proves the lemma. Q.E.D.

It follows that

Proposition 2.2. *Σ is a critical point for F if and only if $H = T^{\perp}$.*

The second variation formula at a critical point is

Theorem 2.3. *Suppose that Σ is a critical point for F . If Σ_s is a compactly supported normal variation of Σ , and*

$$\partial_s \Big|_{s=0} \Sigma_s = V,$$

then setting $F'' = \partial_{ss} \Big|_{s=0} (F(\Sigma_s))$ gives

$$(2.2) \quad F'' = \int_{\Sigma} -\langle V, LV \rangle e^{\langle T, x \rangle} d\mu,$$

where

$$\begin{aligned} LV &= \Delta^{\perp} V + \nabla_{T^{\perp}}^{\perp} V + \langle \langle A, V \rangle, A \rangle \\ &= \left(\Delta V^{\alpha} + \langle T, \nabla V^{\alpha} \rangle + g^{ik} g^{jl} V^{\beta} h_{ij}^{\beta} h_{kl}^{\alpha} \right) e_{\alpha}. \end{aligned}$$

Proof. Letting primes denote derivatives with respect to s at $s = 0$, differentiating (2.1) gives

$$\begin{aligned} F'' &= \int_{\Sigma} \left\{ \frac{\partial}{\partial s} \Big|_{s=0} (\langle T - H, V \rangle) + \langle T^{\perp} - H, V \rangle^2 \Big|_{s=0} \right\} e^{\langle T, x \rangle} \\ (2.3) \quad &= \int_{\Sigma} \{ -\langle H', V \rangle + \langle T - H, V' \rangle \} e^{\langle T, x \rangle}. \end{aligned}$$

Similar to the derivation of the second variation formula for the area, we have

$$(2.4) \quad \langle H', V \rangle = \langle \Delta^{\perp} V + g^{ik} g^{jl} V^{\beta} h_{ij}^{\beta} h_{kl}^{\alpha} e_{\alpha}, V \rangle.$$

On the other hand, since $\langle [V, T^T], V \rangle = 0$, it follows that

$$\begin{aligned} \langle T - H, V' \rangle &= \langle T - H, \overline{\nabla}_V^T V \rangle = \langle T, \overline{\nabla}_V^T V \rangle = \langle T^T, \overline{\nabla}_V V \rangle = -\langle \overline{\nabla}_V T^T, V \rangle \\ (2.5) \quad &= -\langle \overline{\nabla}_{T^T} V, V \rangle = -\langle \nabla_{T^T}^{\perp} V, V \rangle. \end{aligned}$$

Putting (2.4) and (2.5) into (2.3) gives (2.2). This proves the theorem. Q.E.D.

2.2. Properties of \mathcal{L} and L . Notice that $T = T^T + H$, where $T^T = \nabla \langle T, x \rangle$, the linear operator defined by

$$\mathcal{L}v = \Delta v + \langle T, \nabla v \rangle = e^{-\langle T, x \rangle} \operatorname{div}_\Sigma \left(e^{\langle T, x \rangle} \nabla v \right)$$

is self-adjoint in a weighted L^2 space. This follows immediately from Stokes' theorem. More precisely,

Lemma 2.4. *If $\Sigma \subset \mathbf{R}^{n+p}$ is a submanifold of \mathbf{R}^{n+p} , u is a C^1 function with compact support, and v is a C^2 function, then*

$$(2.6) \quad \int_\Sigma u(\mathcal{L}v) e^{\langle T, x \rangle} = - \int_\Sigma \langle \nabla v, \nabla u \rangle e^{\langle T, x \rangle}$$

The next corollary is an extension of Lemma 2.4, which follows immediately by choosing cut-off functions and using the dominated convergence theorem, the same as in the proof of Corollary 3.10 in [4].

Corollary 2.5. *Suppose that $\Sigma \subset \mathbf{R}^{n+p}$ is a complete submanifold of \mathbf{R}^{n+p} without boundary. If u, v are C^2 functions with*

$$\int_\Sigma (|u \nabla v| + |\nabla u| |\nabla v| + |u \mathcal{L}v|) e^{\langle T, x \rangle} < \infty,$$

then we get

$$\int_\Sigma u(\mathcal{L}v) e^{\langle T, x \rangle} = - \int_\Sigma \langle \nabla v, \nabla u \rangle e^{\langle T, x \rangle}.$$

Now we calculate some equalities that we think will be useful in the future.

Proposition 2.6. *If $\Sigma^n \subset \mathbf{R}^{n+p}$ is a translating soliton, then for every constant vector field y ,*

$$(2.7) \quad Ly^\perp = 0.$$

Especially, choosing $y = T$, we have

$$(2.8) \quad LH = 0.$$

Besides, we have

$$(2.9) \quad \mathcal{L}x^A = T^A.$$

Proof. Fix $p \in \Sigma$ and choose an orthonormal frame $\{e_i\}$ such that $\nabla_{e_i} e_j(p) = 0$, $g_{ij} = \delta_{ij}$ in a neighborhood of p . We have

$$(2.10) \quad \nabla_{e_i}^\perp y^\perp = \nabla_{e_i}^\perp (y - \langle y, e_j \rangle e_j) = -\langle y, e_j \rangle h_{ij}^\alpha e_\alpha.$$

Especially, choosing $y = T$, we have

$$(2.11) \quad \nabla_{e_i}^\perp H = \nabla_{e_i}^\perp T^\perp = -\langle T, e_j \rangle h_{ij}^\alpha e_\alpha,$$

i.e.,

$$(2.12) \quad H_{,i}^\alpha = -\langle T, e_j \rangle h_{ij}^\alpha.$$

Taking another covariant derivative at p , it gives

$$(2.13) \quad \begin{aligned} \nabla_{e_k}^\perp \nabla_{e_i}^\perp y^\perp &= -e_k \langle y, e_j \rangle h_{ij}^\alpha e_\alpha - \langle y, e_j \rangle h_{ij,k}^\alpha e_\alpha \\ &= -\langle y, h_{kj}^\beta e_\beta \rangle h_{ij}^\alpha e_\alpha - \langle y, e_j \rangle h_{ik,j}^\alpha e_\alpha, \end{aligned}$$

where we used (2.10), $\nabla_{e_k} e_j(p) = 0$, and the Codazzi equation in the last equality. Taking the trace of (2.13) and using $H = T^\perp$, we conclude that

$$\begin{aligned}\Delta^\perp y^\perp &= -\langle y, h_{ij}^\beta e_\beta \rangle h_{ij}^\alpha e_\alpha - \langle y, e_j \rangle H_{,j}^\alpha e_\alpha \\ &= -y^\beta h_{ij}^\beta h_{ij}^\alpha e_\alpha + \langle y, e_j \rangle \langle T, e_i \rangle h_{ij}^\alpha e_\alpha \\ &= -y^\beta h_{ij}^\beta h_{ij}^\alpha e_\alpha - \langle T, e_i \rangle \nabla_{e_i}^\perp y^\perp \\ &= -y^\beta h_{ij}^\beta h_{ij}^\alpha e_\alpha - \nabla_{T^T}^\perp y^\perp.\end{aligned}$$

This proves (2.7).

Since $\Delta x = H$ and $H = T^\perp$, we have

$$\Delta x^A = \langle H, E_A \rangle = \langle T^\perp, E_A \rangle = \langle T, E_A^\perp \rangle = \langle T, E_A \rangle - \langle T, E_A^T \rangle = T^A - \langle T, E_A^T \rangle.$$

Hence

$$\mathcal{L}x^A = \Delta x^A + \langle T, \nabla x^A \rangle = \Delta x^A + \langle T, (E_A)^T \rangle = T^A,$$

This proves (2.9).

Q.E.D.

2.3. Hamiltonian L-stability. In this subsection, we will define Hamiltonian L-stability of Lagrangian translating solitons. Recall the definition of Hamiltonian variations on a Lagrangian submanifold.

Definition 2.1. [10] *Let $(M, \bar{\omega})$ be a symplectic manifold M . Let $\Sigma \subset M$ be a Lagrangian submanifold and V be a vector field along Σ . V is called a Hamiltonian variation if it satisfies that the one form $i^*(V] \bar{\omega})$ on Σ is exact.*

The Hamiltonian variation has an equivalent definition.

Lemma 2.7. [10] *A normal variation V on Σ is Hamiltonian if and only if*

$$V = J \nabla f,$$

where f is a function on Σ and ∇ is the gradient on Σ with respect to the induced metric.

Now we are ready to define Hamiltonian L-stability of Lagrangian translating solitons.

Definition 2.2. *We say a Lagrangian translating soliton Σ is Hamiltonian L-stable if for every compactly supported Hamiltonian variations Σ_s with $\Sigma_0 = \Sigma$, $F'' = \int_\Sigma -\langle V, LV \rangle e^{\langle T, x \rangle} \geq 0$.*

3. PROOF OF THE MAIN THEOREM

Note that the normal bundle brings much difficulty to the study of L-stability of translating solitons in the general higher codimension case. However, in [10], Oh studied Hamiltonian stability of minimal Lagrangian submanifolds in Kähler-Einstein manifolds, and characterized Hamiltonian stability by a condition on the first eigenvalue of Δ acting on functions. The key point of Oh's proof is that, for a minimal Lagrangian submanifold of a Kähler-Einstein manifold, the set of Hamiltonian variations is an invariant subspace of the Jacobi operator. This idea was then used to study Hamiltonian F-stability of Lagrangian self-shrinkers ([9], [12]). It is natural to think that this property also holds for Lagrangian translating solitons. This property inspired us to show the following equality, which well characterizes how the operator L acts on Hamiltonian variations.

Theorem 3.1. *Suppose $\Sigma^n \subset \mathbf{C}^n$ is a Lagrangian translating soliton. Then for every function f on Σ ,*

$$(3.1) \quad LJ\nabla f = J\nabla \mathcal{L}f.$$

This implies that the set of Hamiltonian variations is an invariant subspace of the operator L .

Proof. Fix a point p . We choose a local orthonormal basis $\{e_i\}_{i=1}^n$ of $T\Sigma$ such that $\nabla_{e_i} e_j(p) = 0$. Then since Σ is Lagrangian, $\{e_{n+i} = Je_i\}_{i=1}^n$ is a local orthonormal basis of $N\Sigma$. In the following we compute at the point p . It is easy to compute that

$$\begin{aligned} LJ\nabla f &= \Delta^\perp(J\nabla f) + \nabla_{T^\perp}^\perp(J\nabla f) + h_{il}^{n+k} f_k h_{il}^{n+j} Je_j \\ &= \left(f_{jii} + \langle T, e_k \rangle f_{jk} + f_k h_{ik}^{n+l} h_{ij}^{n+l} \right) Je_j, \end{aligned}$$

where in the last equality we used the Lagrangian property $h_{il}^{n+k} = h_{ik}^{n+l}$. On the other hand,

$$\begin{aligned} J\nabla \mathcal{L}f &= J\nabla (\Delta f + T^T f) \\ &= f_{iij} Je_j + J\nabla \langle T^T, \nabla f \rangle \\ &= (f_{iji} - f_i R_{jki} + e_j \langle T^T, \nabla f \rangle) Je_j \\ &= \left(f_{jii} - f_i h_{ij}^{n+l} h_{kk}^{n+l} + f_i h_{jk}^{n+l} h_{ik}^{n+l} + \langle \nabla_{e_j} T^T, \nabla f \rangle + \langle T^T, \nabla_{e_j} \nabla f \rangle \right) Je_j \\ &= \left(f_{jii} - f_i h_{ij}^{n+l} \langle T^\perp, e_{n+l} \rangle + f_k h_{ik}^{n+l} h_{ij}^{n+l} + \langle \bar{\nabla}_{e_j} T, \nabla f \rangle - \langle \bar{\nabla}_{e_j} T^\perp, \nabla f \rangle \right. \\ &\quad \left. + f_{jk} \langle T, e_k \rangle \right) Je_j \\ &= \left(f_{jii} - f_i h_{ij}^{n+l} \langle T^\perp, e_{n+l} \rangle + f_k h_{ik}^{n+l} h_{ij}^{n+l} + \left\langle T^\perp, \bar{\nabla}_{e_j} (f_k e_k) \right\rangle + \langle T, e_k \rangle f_{jk} \right) Je_j \\ &= \left(f_{jii} - f_i h_{ij}^{n+l} \langle T^\perp, e_{n+l} \rangle + f_k h_{ik}^{n+l} h_{ij}^{n+l} + f_k \langle T^\perp, h_{jk}^{n+l} e_{n+l} \rangle + \langle T, e_k \rangle f_{jk} \right) Je_j \\ &= \left(f_{jii} + \langle T, e_k \rangle f_{jk} + f_k h_{ik}^{n+l} h_{ij}^{n+l} \right) Je_j, \end{aligned}$$

where in the third equality we used the Ricci formula; in the fourth equality we used the Gauss equation; and in the fifth equality we used the translating soliton equation $H = T^\perp$. This proves the theorem. Q.E.D.

Now we recall our main theorem.

Theorem 3.2. *Any Lagrangian translating soliton is Hamiltonian L -stable.*

Proof. Recall that the second variation formula for F is

$$(3.2) \quad F'' = \int_{\Sigma} -\langle V, LV \rangle e^{\langle T, x \rangle}.$$

Now Assume V is a compactly supported Hamiltonian variation, then there exists a function f , such that $V = J\nabla f$. Putting it into (3.2), and using (3.1), we have

$$(3.3) \quad F'' = \int_{\Sigma} -\langle J\nabla f, LJ\nabla f \rangle e^{\langle T, x \rangle} = \int_{\Sigma} -\langle J\nabla f, J\nabla \mathcal{L}f \rangle e^{\langle T, x \rangle} = \int_{\Sigma} (\mathcal{L}f)^2 e^{\langle T, x \rangle} \geq 0,$$

where the last equality used Lemma 2.4 and the fact that $V = J\nabla f$ is compactly supported implies that $\mathcal{L}f$ is compactly supported. This proves the theorem. Q.E.D.

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